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Global Identities in the Non-normal Newton–Padé Approximation Table

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This note is concerned with the Newton–Padé table containing rational interpolants with varying numerator and denominator degrees. In the general case some entries of the table can be equal, combined in so-called singular blocks. Any singular block in this non-normal Newton–Padé approximation table consists of squares forming a symmetric tail. It is the aim of this note to present global identities between neighboring entries of a singular block. In particular, we generalize Cordellier's identities for Padé approximation. The resulting algorithmic aspects, e.g., a reliable modification of Claessens' cross-rule (G. Claessens, *Numer. Math.* **29** (1978), 227–231), are discussed in (B. Beckermann and C. Carstensen, *Numer. Algorithms* **3** (1992), 29–44). © 1993 Academic Press, Inc.

1. THE NEWTON–PADÉ APPROXIMATION TABLE: PRELIMINARIES

Let $(z_i)_{i \in \mathbb{N}_0}$ be (not necessarily distinct) knots in the complex plane $\mathbb{C}$ and let f be a function which is sufficiently smooth in a neighborhood of these knots. Then, the *rational interpolation problem* (sometimes called *Hermite* or *osculatory* rational interpolation problem) reads: for any $m, n \in \mathbb{N}_0$ find polynomials $p_{m,n}$ and $q_{m,n}$ (called solutions of the (m, n)-rational interpolation problem) of degree m and n, respectively, such that $f - p/q$ has the zeros $z_0, z_1, ..., z_{m+n}$ counting multiplicities (abbr.: c.m.).

By [10, 12] this problem is closely connected with the *Newton–Padé approximation problem* which is also under consideration here. As Claessens pointed out (cf. [4, Theorem 1]) there exist unique polynomials $p^*_{m,n}$ and $q^*_{m,n}$, $q^*_{m,n}$ being a monic polynomial, of "minimal degree"

$$\deg p^*_{m,n} \leqslant m \qquad \text{and} \qquad \deg q^*_{m,n} \leqslant n$$

which satisfy the *interpolation conditions*

$$(p^*_{m,n} - f \cdot q^*_{m,n}) \qquad \text{has the zeros } z_0, ..., z_{m+n} \quad \text{c.m.}$$

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such that other solutions (of the so-called (m, n)-Newton–Padé approximation problem) are of the form $s \cdot p^*_{m,n}$, $s \cdot q^*_{m,n}$, where s is a polynomial of degree less than or equal to $\min\{m - \deg p^*_{m,n}, n - \deg q^*_{m,n}\}$. $p^*_{m,n}$, $q^*_{m,n}$ is the *minimal solution* of the (m, n)-Newton–Padé approximation problem.

Note that, consequently, two solutions of the (m, n)-Newton–Padé approximation problem have the same reduced form which is the meromorphic function

$$r_{m,n} := \frac{p^*_{m,n}}{q^*_{m,n}}.$$

Therefore, $r_{m,n}$ is called *the solution* of the (m, n)-Newton–Padé approximation problem. Moreover, $(r_{m,n} \mid m, n \in \mathbb{N}_0)$ is called the *Newton–Padé approximation table.*

The computation of some entries of the Newton–Padé approximation table can be done recursively by various algorithms like, e.g., Claessens' cross-rule [6] (see (4) below). Unfortunately, Claessens' cross-rule fails if certain neighbouring values of the Newton–Padé approximation table are equal to each other; such entries will be united and called a *singular block*, while the Newton–Padé approximation table is called *non-normal.*

We note that the notation of *normality* is not uniquely used in the literature (cf., e.g., [4, p. 156] for paranormality).

The paper is organized as follows. In Section 2 the structure of the non-normal Newton–Padé table is recalled from [3–5, 7, 8]. We state a general cross-rule type identity connecting all neighbors of a singular block in Section 3, where some illustrating examples are also given. In addition, we classify those entries which already determine the whole neighborhood of a singular block. Proofs are given in Section 4. In Section 5 we consider the particular square block case which leads to identities of the Cordellier type. The connections between the general case and the square block obtained by reordering of interpolation knots [8, 12] are discussed in Section 6. Finally, in Section 7 we study more detailed consequences of the cross-rule type identity of Section 3. A couple of examples are given concerning the relationships between neighboring entries which strongly depend on the shape of the singular block in the non-normal Newton–Padé table.

2. The Non-normal Case: Notations

As Claessens pointed out in [5], any singular block consists of one or more square blocks which are overlapping along a common diagonal forming a "symmetric tail," similar geometric structures have been observed in [2] for a more general approximation problem. In the sequel, we describe more detailed the form of singular blocks in the Newton–Padé table, we refer to [3–5, 7, 8] for proofs and further explanations.

Let $(m, n) \in \mathbb{N}_0^2$ be the starting point of the (finite) singular block which consists of squares forming a symmetric tail, i.e., there exist $p \in \mathbb{N}$ and $0 = k_0 = k_{2p} = l_0$, $l_{2p} = p$ and $k_1, ..., k_{2p-1} \in \mathbb{N}$, $l_1, ..., l_{2p-1} \in \mathbb{N}_0$ such that for any $j \in \{0, ..., 2p\}$

$$C := r_{m,n} = r_{m+l_j+k_j-1-i,\, n+l_j+i} \qquad \text{for any} \quad i \in \{0, 1, ..., k_j - 1\},$$

while

$$SW_j := r_{m+l_j+k_j,\, n+l_j-1} \neq C \neq r_{m+l_j-1,\, n+l_j+k_j} =: NE_j.$$

Moreover $k_{j+1} - k_j \in \{\pm 1\}$, $l_{j+1} - l_j \in \{0, 1\}$ and $j = k_j + 2l_j$ for any $j \in \{0, ..., 2p\}$. Define

$$A = \{j \in \{0, ..., 2p-1\} \mid k_{j+1} = k_j + 1\} \qquad \text{and} \qquad U = \{0, ..., 2p-1\} \backslash A$$

and notice that card $A = p =$ card U. The case of an infinite singular block $(p = \infty)$ can be described similarly.

Remarks. (i) The polynomials of the *minimal solution* $p^*_{m,n}$ and $q^*_{m,n}$ are not necessarily irreducible but common factors have zeros from $z_0, ..., z_{m+n}$ which are then called *unattainable points*; cf. [4, Theorem 2]. It is known and easily seen that, in the present notations, z_{m+n+j} is attainable iff $j \in A$. Consequently, for any $j \in \{1, ..., 2p\}$, $i = \{0, 1, ..., k_j - 1\}$, $z \in \mathbb{C}$

$$p^*_{m+l_j+k_j-1-i,\, n+l_j+i}(z) = p^*_{m,n}(z) \cdot \prod_{v \in U,\, v < j} (z - z_{m+n+v}) \tag{1}$$

$$q^*_{m+l_j+k_j-1-i,\, n+l_j+i}(z) = q^*_{m,n}(z) \cdot \prod_{v \in U,\, v < j} (z - z_{m+n+v}). \tag{2}$$

This justifies that U describes the unattainable points.

(ii) We note that $i \in A$, $j \in U$, $j < i$ implies $z_{m+n+i} \neq z_{m+n+j}$. (For a proof of (i), (ii) see, e.g., [5, 12]). This will be frequently used throughout the sequel where we will say carelessly that "an unattainable point cannot become attainable later."

(iii) In the present notations, k_j denotes the number of equal approximants, l_j is the number of unattainable and $k_j + l_j$ the number of attainable points (related to the given singular block) of approximants of the antidiagonal no. $m + n + j - 1$. Hence in the case $j \in A$, $j \in U$, locally the block becomes "wider," and "narrower," respectively (compare [8, p. 555]).

TABLE I

$r_{m,n}$	$n=0$	1	2	3	4	5
$m=0$	1/2	NE_0	NE_1	NE_2	NE_3	
1	SW_0	C	C	C	NE_4	
2	SW_1	C	C	C	NE_5	NE_6
3	SW_2	C	C	C	C	NE_7
4	SW_3	SW_4	SW_5	C	C	NE_8
5			SW_6	SW_7	SW_8	

EXAMPLE 1. As in [5, Example 2], define $z_{0+4i} := -3$, $z_{1+4i} := 0$, $z_{2+4i} := 1$, $z_{3+4i} := 2$, $z_{12+i} := 3$ for $i = 0, 1, 2$ and $f(-3) = 1/2$, $f(0) = 2$, $f(1) = 3/2$, $f(2) = 4/3$, $f'(-3) = -1/4$, $f'(0) = 1$, $f'(1) = 1$, $f'(2) = -1/9$, $f''(-3) = 1$, $f''(0) = 2$, $f''(1) = 1$, $f''(2) = 2/27$, $f(3) = 5/4$, $f'(3) = 1$, $f''(3) = 1$. Then we have $C(z) = (z+2)/(z+1)$, $z \in \mathbb{C}$, and the structure is shown in Table I (cf. [5, Table 1]).

In this example we have $p = 4$, $(k_0, ..., k_8) = (0, 1, 2, 3, 2, 1, 2, 1, 0)$, $(l_0, ..., l_8) = (0, 0, 0, 0, 1, 2, 2, 3, 4)$, $A = \{0, 1, 2, 5\}$ and $U = \{3, 4, 6, 7\}$.

Remark. (iv) For Padé approximants the mnemonic S_j, N_j, E_j of compass points is frequently used to describe the neighbors of the singular block. For Newton–Padé approximants, this notation would be as follows;

$$NE_j = \begin{cases} N_j & \text{if } j \in A \text{ and } j-1 \notin U \\ E_j & \text{if } j-1 \in U \end{cases},$$

$$SW_j = \begin{cases} W_j & \text{if } j \in A \text{ and } j-1 \notin U \\ S_j & \text{if } j-1 \in U \end{cases}.$$

Since N_j, S_j, W_j can be computed e.g., using Claessens' cross-rule (if no other singular block occurs there), throughout this note, we are mainly interested in certain rules determining E_j. Consequently, the neighbors in "outer corners," i.e., NE_j, SW_j with $j \in U$ and $j-1 \in A$, are neither given data nor determinated by some rule. In the example this is true for $j = 3$, $j = 6$.

For any $j \in \{0, 1, ..., 2p\}$ let the meromorphic function a_j be defined by

$$\frac{1}{SW_j - C} - \frac{1}{NE_j - C} = a_j \cdot R \cdot \Omega_j, \tag{3}$$

where, for any $z \in \mathbb{C}$,

$$\Omega_j(z) := \frac{\prod_{i \in U, i<j} (z - z_{m+n+i})}{\prod_{i \in A, i<j} (z - z_{m+n+i})}$$

$$R(z) := \frac{q^*_{m,n}(z)^2}{a_{m,n} \cdot \omega_{0,m+n}(z)}$$

$$\omega_{\mu,\nu}(z) := \prod_{i=\mu}^{\nu-1} (z - z_i)$$

and, by convention, empty products are 1. Here, $a_{m,n}$ is the leading coefficient of $p^*_{m,n}$ (the leading coefficient of $q^*_{m,n}$ is 1). Since $SW_0 \neq C \neq NE_0$ we have $a_{m,n} \neq 0$; cf. Lemma 1 below: $\deg p^*_{m,n} = m$, $\deg q^*_{m,n} = n$.

Note that, by (3), any recurrence relation for a_j leads to an identity in the non-normal Newton–Padé approximation table and vice versa. Thus we will have a closer look at the a_j which are in fact monic polynomials of degree k_j (cf. Lemma 2).

3. Motivation and First Results

We pause in treating the general case and assume the normal case, i.e., $p = 1$, first. Then *Claessens' cross-rule* [6] (generalizing *Wynn's identity*, see, e.g., [1], for Padé approximants) gives

$$\frac{1}{z - z_{m+n}} \cdot \left\{ \frac{1}{SW_0(z) - C(z)} - \frac{1}{NE_0(z) - C(z)} \right\}$$
$$= \frac{1}{z - z_{m+n+1}} \cdot \left\{ \frac{1}{SW_2(z) - C(z)} - \frac{1}{NE_2(z) - C(z)} \right\}, \qquad (4)$$

$z \in \mathbb{C}$, which, by (3), can be written as

$$a_0 \cdot \Omega_1 = a_2 \cdot \Omega_1.$$

The first aim of this note is to adapt this identity to singular blocks. Assume, for the moment, that the Newton–Padé approximation table is normal, i.e. its entries can be computed by Claessens' cross-rule. Next, we take all cross-rules with center C in

$$\{(m + l_j + k_j - 1 - i, n + l_j + i) \mid j = 0, ..., 2p, i = 0, ..., k_j - 1\},$$

which is later an element of the singular block; cf. Table 1. Then we add all these cross-rules and observe that all the "inner differences" (i.e., a term where two approximants of the form $r_{m+l_j+k_j-1-i, n+l_j+i}$ with

$j \in \{1, ..., 2p-1\}$ and $i \in \{0, ..., k_j - 1\}$ arise) occur twice and their sum vanishes. Hence we obtain the *composed Claessens' identity*

$$\sum_{j \in A} \frac{1}{z - z_{m+n+j}} \cdot \left\{ \frac{1}{SW_j(z) - r_{m+l_{j+1}+k_{j+1}-1,\, n+l_{j+1}}(z)} - \frac{1}{NE_j(z) - r_{m+l_{j+1},\, n+l_{j+1}+k_{j+1}-1}(z)} \right\}$$
$$= \sum_{j-1 \in U} \frac{1}{z - z_{m+n+j-1}} \cdot \left\{ \frac{1}{SW_j(z) - r_{m+l_{j-1}+k_{j-1}-1,\, n+l_{j-1}}(z)} - \frac{1}{NE_j(z) - r_{m+l_{j-1},\, n+l_{j-1}+k_{j-1}-1}}(z) \right\}, \tag{5}$$

$z \in \mathbb{C}$. Now consider the non-normal case. It is not hard to see that by arbitrarily small perturbations of the data we obtain a normal situation such that (5) holds. Consequently, by continuity, we expect that we can replace each term in any denominator of (5) by $C(z)$. Then, (3) gives

$$\sum_{j \in A} a_j(z) \cdot \Omega_{j+1}(z) = \sum_{j \in U} a_{j+1}(z) \cdot \Omega_j(z), \qquad z \in \mathbb{C}. \tag{6}$$

Since in (6) no denominator vanishes in the case of a singular block, we may hope (and will prove in the sequel) that (6) holds in the situation of the previous section.

Note that (6) gives only one singular rule, although we are interested in at least p identities of such type. Our first result implies (6) and covers this question.

THEOREM 1. *For all $t, z \in \mathbb{C}$ there holds*

$$\sum_{j \in A} a_j(z) \cdot \Omega_{j+1}(t) = \sum_{j \in U} a_{j+1}(z) \cdot \Omega_j(t), \tag{7}$$

where a_j and Ω_j are defined in (3). *This equation contains at least p identities connecting approximants on the boundary of the singular block.*

Using the classical notations as in (4), Eq. (7) takes the from

$$\sum_{j \in A} \frac{1}{t - z_{m+n+j}} \cdot \left\{ \frac{1}{SW_j(z) - C(z)} - \frac{1}{NE_j(z) - C(z)} \right\} \cdot \prod_{i \in U,\, i<j} \frac{t - z_{m+n+i}}{z - z_{m+n+i}} \cdot \prod_{i \in A,\, i<j} \frac{z - z_{m+n+i}}{t - z_{m+n+i}}$$
$$= \sum_{j-1 \in U} \frac{1}{t - z_{m+n+j-1}} \cdot \left\{ \frac{1}{SW_j(z) - C(z)} - \frac{1}{NE_j(z) - C(z)} \right\} \cdot \sum_{i \in U,\, i<j} \frac{t - z_{m+n+i}}{z - z_{m+n+i}} \prod_{i \in A,\, i<j} \frac{z - z_{m+n+i}}{t - z_{m+n+i}}. \tag{8}$$

Theorem 1 is proved in the next section, but we will already discuss a first application of the general identity (7). Multiplication of (7) with $\prod_{i \in A} (t - z_{m+n+i})$ leads to

$$\sum_{\substack{j \in A \\ j-1 \notin U}} a_j(z) \cdot w_{j+1}(t) = \sum_{\substack{j \in U \\ j+1 \notin A}} a_{j+1}(z) \cdot w_j(t) + \sum_{\substack{j \in U \\ j+1 \in A}} a_{j+1}(z) \cdot (w_j(t) - w_{j+2}(t)), \tag{9}$$

where

$$\begin{aligned} w_j(t) &:= \prod_{i \in U, i < j} (t - z_{m+n+i}) \cdot \prod_{i \in A, i \geq j} (t - z_{m+n+i}) \\ &= \Omega_j(t) \cdot \prod_{i \in A} (t - z_{m+n+i}). \end{aligned} \tag{10}$$

In addition, let

$$W_j(t) := \prod_{i \in U, i < j} (t - z_{m+n+i}) \cdot \prod_{i \in A, i \geq j+2} (t - z_{m+n+i}) \tag{11}$$

and note that

$$\begin{aligned} &j \in U, j+1 \in A \Rightarrow w_j(t) - w_{j+2}(t) = (z_{m+n+j} - z_{m+n+j+1}) \cdot W_j(t) \\ &j \in U, j+1 \notin A \Rightarrow w_j(t) = W_j(t). \end{aligned}$$

As mentioned in Remark (ii), for $j \in U$, $j+1 \in A$, the unattainable point z_{m+n+j} cannot become attainable, in particular (since $z_{m+n+j+1}$ is attainable) $z_{m+n+j} \neq z_{m+n+j+1}$. In Lemma 3 (see the next section) we will prove that $(W_j \mid j \in U)$ are linearly independent.

This can be used as follows; see also Example 3 below. Provided that $z \in \mathbb{C}$ is fixed and that $a_j(z)$ is known for all $j \in A$, $j-1 \notin U$, the left hand side of Eq. (9), denoted by $h(t)$, is known too. Then, (9) proves that h lies in the linear hull of $(W_j \mid j \in U)$ and Lemma 3 implies that the coefficients and hence $(a_j(z) \mid j-1 \in U)$ are uniquely defined and can explicity be computed from (9).

EXAMPLE 2. Let us consider the Padé approximation table which is included as a particular case for confluent knots, i.e., $0 = z_0 = z_1 = z_2 = \cdots$. We mentioned that a knot which once has become unattainable cannot be an attainable point anymore; cf. Remark (ii) in Section 2. Since we have exactly one knot 0, we claim the well-known fact that singular blocks are squares for Padé approximations, i.e., $A = \{0, 1, ..., p-1\}$ and $U = \{p, ..., 2p-1\}$.

Then, by definition, (8) gives for any $t, z \in \mathbb{C}$

$$\sum_{j=0}^{p-1} \frac{z^j}{t^{j+1}} \cdot \left\{ \frac{1}{SW_j(z) - C(z)} - \frac{1}{NE_j(z) - C(z)} \right\}$$
$$= \sum_{j=p+1}^{2p} \frac{z^{2p-j}}{t^{2p-j+1}} \cdot \left\{ \frac{1}{SW_j(z) - C(z)} - \frac{1}{NE_j(z) - C(z)} \right\}.$$

This yields Cordellier's identities for $j \in \{p+1, ..., 2p\}$

$$\frac{1}{SW_j(z) - C(z)} - \frac{1}{NE_j(z) - C(z)} = \frac{1}{SW_{2p-j}(z) - C(z)} - \frac{1}{NE_{2p-j}(z) - C(z)}.$$

EXAMPLE 3. We continue Example 1 from [5, Example 2]. The neighbors, defined in Table I, are given in Table II. Using $R(z) = (z+1)^2/(z(z+3))$ and $C(z) = (z+2)/(z+1)$, the polynomials a_j can be computed as shown in Table II. We assume that $(a_j \mid j \in A$ and $j-1 \notin U)$ is known from Table II. By (8) and (11), (9) reads

$$\begin{aligned}(t-2)^2 (t+3) + (z-1)(t-2)(t+3) + z(z-3)(t-2) \\ = a_4(z)(t-2) + a_7(z)\, t(t-1) + a_8(z)\, t(t-1)(t+3) \\ + a_5(z)\{t(t-2) - t(t-1)\},\end{aligned}$$

whence

$$\begin{aligned}t^3 + t^2(z-2) - t(z^2 - 2z - 9) - 2(z^2 - 9) \\ = a_8(z)\, t^3 + t^2(a_7(z) + 2a_8(z)) \\ + t(a_4(z) - a_5(z) - a_7(z) - 3a_8(z)) - 2a_4(z).\end{aligned}$$

Comparing coefficients one easily computes $(a_j \mid j-1 \in U)$, as shown in Table II.

As mentioned above, a_3 and a_6, i.e., $(a_j \mid j \in U$ and $j-1 \in A)$, do note occur in these calculations as well as throughout this note.

Following the ideas of Example 3, we see that, given $(a_j(z) \mid j \in A$ and $j-1 \notin U)$, we are able to determine $(a_j(z) \mid j-1 \in U)$. The next result, proved at the end of Section 6, shows that even less input data are sufficient (for non-square blocks).

THEOREM 2. *Assume that the knots* $z_{m+n}, z_{m+n+1}, ..., z_{m+n+2p+1}$ *and the sets* A *and* U *are known. Then, using the values*

$$(a_j(z) \mid j \in A, k_j > \max\{k_i \mid i = 0, ..., j-1 \text{ and } i \in A\}), \tag{12}$$

the quantities $(a_j(z) \mid j \in A$ *or* $j-1 \in U)$ *can be computed.*

TABLE II

j	$NE_j(z)$	$SW_j(z)$	$a_j(z)$	$w_j(t)$
0	$\dfrac{-2}{z-1}$	$\dfrac{z+2}{2}$	1	$(t-1)(t-2)^2(t+3)$
1	$\dfrac{6}{z^2+3}$	$\dfrac{-z^2-z+8}{4}$	$z-1$	$(t-2)^2(t+3)$
2	$\dfrac{-24}{z^3-2z^2-3z-12}$	$\dfrac{z^3-z^2-6z+24}{12}$	z^2-3z	$(t-2)(t+3)$
3	$\dfrac{-24}{z^4+z^3-9z^2+3z+12}$	$\dfrac{z^4+2z^3-9z^2-6z+48}{24}$	$z^3-9z+12$	$t-2$
4	$\dfrac{3z-18}{z^4+z^3-9z^2+6z-9}$	$\dfrac{-z^4-2z^3+9z^2-21z+18}{15z-9}$	z^2-9	$t(t-2)$
5	$\dfrac{41z^2+97z-90}{5z^4+5z^3-4z^2+71z-45}$	$\dfrac{-5z^4-10z^3+46z^2+107z-90}{z^2+76z-45}$	$z+1$	$t(t-1)(t-2)$
6	$\dfrac{-1066z^2-8282z+7380}{205z^5-205z^4-2255z^3+3239z^2-5986z+3690}$	$\dfrac{-5z^5+65z^3+14z^2-302z+180}{74z^2-196z+90}$	z^2-z-10	$t(t-1)$
7	$\dfrac{-4z^3-30z^2-194z+180}{5z^5-5z^4-59z^3+79z^2-142z+90}$	$\dfrac{-5z^5+71z^3+20z^2-314z+180}{6z^3+74z^2-202z+90}$	$z-4$	$t(t-1)(t+3)$
8	$\dfrac{68z^4+280z^3-22z^2+990z+540}{15z^5+53z^4+27z^3+169z^2-630z+270}$	$\dfrac{-15z^5+8z^4+245z^3+68z^2-990z+540}{8z^4+42z^3+214z^2-630z+270}$	1	$t^2(t-1)(t+3)$

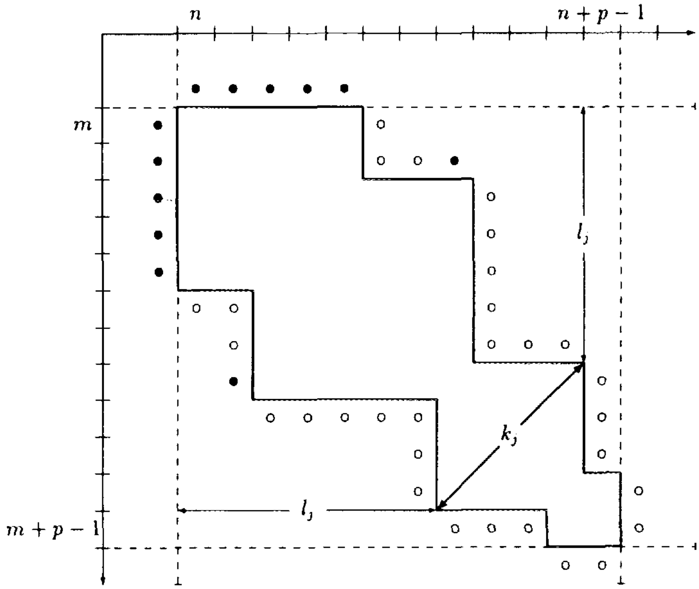

FIGURE 1

The assertion of Theorem 2 is illustrated in Fig. 1, where in the non-normal Newton–Padé table the singular block is hatched. The positions of approximants which have to be known are marked by "●." On the other hand "○" designates the position of SW_j or NE_j (NE_j and SW_j are the upper and lower neighbor of the singular block in the antidiagonal no. $m+n+j-1$) which are related to $a_j(z)$. Then $a_j(z)$ can be computed using the entries which are marked by "●" in previous antidiagonals.

4. PROOFS

We prove Theorem 1 with the aid of two lemmata. The first lemma is already known for normal Newton–Padé tables [11].

LEMMA 1. [3, 11]. *For any* $m, n \in \mathbb{N}_0$

$$r_{m+1,n} - r_{m,n} = \begin{cases} 0 & \text{iff } \deg p^*_{m+1,n} < m+1 \\ & \text{or } \deg q^*_{m,n} < n, \\ a_{m+1,n} \cdot \dfrac{\omega_{0,m+n+1}}{q^*_{m,n} \cdot q^*_{m+1,n}} & \text{otherwise} \end{cases} \tag{13}$$

$$r_{m,n} - r_{m,n+1} = \begin{cases} 0 & \textit{iff} \quad \deg p^*_{m,n} < m \\ & \textit{or} \quad \deg q^*_{m,n+1} < n+1, \\ a_{m,n} \cdot \dfrac{\omega_{0,m+n+1}}{q^*_{m,n} \cdot q^*_{m,n+1}} & \textit{otherwise} \end{cases} \tag{14}$$

$$r_{m+1,n} - r_{m,n+1} = \begin{cases} 0 & \textit{iff} \quad \deg p^*_{m+1,n} < m+1 \\ & \textit{or} \quad \deg q^*_{m,n+1} < n+1. \\ a_{m+1,n} \cdot \dfrac{\omega_{0,m+n+2}}{q^*_{m+1,n} \cdot q^*_{m,n+1}} & \textit{otherwise} \end{cases} \tag{15}$$

LEMMA 2 (cf. [3]). *$a_0 = 1 = a_{2p}$ and for any $j \in \{0, 1, ..., 2p-1\}$, there holds*

$$a_{j+1} = a_j \cdot \omega_{m+n+j,m+n+j+1} - c_j \qquad \textit{if} \quad j \in A,$$
$$a_j = a_{j+1} \cdot \omega_{m+n+j,m+n+j+1} - c_{j+1} \qquad \textit{if} \quad j \in U,$$

where

$$c_j = \frac{a_{m,n}}{a_{m+l_j+k_j,n+l_j-1}} + \frac{a_{m+l_j-1,n+l_j+k_j}}{a_{m,n}} \in \mathbb{C},$$

and the first and second term must be neglected if $SW_{j+1} = SW_j$ and $NE_{j+1} = NE_j$, respectively. In particular, $a_0, ..., a_{2p}$ are monic polynomials of degree $k_0, ..., k_{2p} \leqslant p$, respectively.

Proof. If $j \in A$ then, by definition

$$\begin{aligned} a_{j+1} &- a_j \cdot \omega_{m+n+j,m+n+j+1} \\ &= \frac{1}{R \cdot \Omega_{j+1}} \cdot \left\{ \frac{1}{SW_{j+1} - C} - \frac{1}{NE_{j+1} - C} - \frac{1}{SW_j - C} + \frac{1}{NE_j - C} \right\} \\ &= \frac{1}{R \cdot \Omega_{j+1}} \cdot \left\{ \frac{NE_{j+1} - NE_j}{(NE_{j+1} - C) \cdot (NE_j - C)} - \frac{SW_{j+1} - SW_j}{(SW_{j+1} - C) \cdot (SW_j - C)} \right\}. \end{aligned}$$

By Lemma 1 any of the appearing differences can be written as the right hand side of Eq. (13), (14), and (15), respectively. For instance,

$$\begin{aligned} &\frac{SW_{j+1} - SW_j}{(SW_{j+1} - C) \cdot (SW_j - C)} \\ &\quad = \frac{r_{m+l_j+k_j+1,n+l_j-1} - r_{m+l_j+k_j,n+l_j-1}}{(r_{m+l_j+k_j+1,n+l_j-1} - r_{m+l_j+k_j,n+l_j}) \cdot (r_{m+l_j+k_j,n+l_j-1} - r_{m+l_j+k_j,n+l_j})} \end{aligned}$$

$$= \frac{a_{m+l_j+k_j+1,n+l_j-1} \cdot \dfrac{\omega_{0,m+n+j}}{q^*_{m+l_j+k_j,n+l_j-1} \cdot q^*_{m+l_j+k_j+1,n+l_j-1}}}{\dfrac{a_{m+l_j+k_j+1,n+l_j-1} \cdot \omega_{0,m+n+j+1}}{q^*_{m+l_j+k_j+1,n+l_j-1} \cdot q^*_{m+l_j+k_j,n+l_j}} \cdot \dfrac{a_{m+l_j+k_j,n+l_j} \cdot \omega_{0,m+n+j}}{q^*_{m+l_j+k_j,n+l_j-1} \cdot q^*_{m+l_j+k_j,n+l_j}}}$$

$$= \frac{q^{*\,2}_{m+l_j+k_j,n+l_j}}{a_{m+l_j+k_j,n+l_j-1} \cdot \omega_{0,m+n+j+1}}$$

$$= \frac{a_{m,n}}{a_{m+l_j+k_j,n+l_j-1}} \cdot R \cdot \Omega_{j+1},$$

where we used (2) for $q^*_{m+l_j+k_j,n+l_j}$.

This proves the theorem if $j \in A$. The proof for $j \in U$ as well as the proof of $a_0 = 1 = a_{2p}$ is similar. ▮

Remark. Note that $a_0 = 1 = a_{2p}$ yields Claessens' cross-rule (4) for $p = 1$. The above "local relations" between the monic polynomials $a_0, ..., a_{2p}$ can be used for a recursive algorithm to compute values of the Newton–Padé approximation table even in the non-normal case.

THEOREM 3. *For any $j \in \{0, ..., 2p\}$ and $t, z \in \mathbb{C}$ there holds*

$$\Omega_j(t) \cdot [t, z]\, a_j = \sum_{i \in A,\, i<j} a_i(z) \cdot \Omega_{i+1}(t) - \sum_{i \in U,\, i<j} a_{i+1}(z) \cdot \Omega_i(t), \tag{16}$$

where $[t, z]$ denotes the divided difference with respect to the knots t, z.

Proof of Theorem 1. Since $a_{2p} = 1$ and $[t, z]\, a_{2p} = 0$, the assertion follows from Theorem 3 with $j = 2p$. ▮

Proof of Theorem 3. By Lemma 2 using $\Omega_{i+1}(z)/\Omega_i(z) = 1/(z - z_{m+n+i})$ and $(z - z_{m+n+i})$ if $i \in A$ and $i \in U$, respectively, we obtain for $t, z \in \mathbb{C}$, $i \in \{0, ..., 2p-1\}$,

$$a_{i+1}(t) \cdot \Omega_{i+1}(t) - a_i(t) \cdot \Omega_i(t) = \begin{cases} -c_i \cdot \Omega_{i+1}(t) & \text{if } i \in A \\ c_{i+1} \cdot \Omega_i(t) & \text{if } i \in U \end{cases} \tag{17}$$

and conversely

$$c_i = a_i(z) \cdot (z - t) + a_i(z) \cdot \frac{\Omega_i(t)}{\Omega_{i+1}(t)} - a_{i+1}(z) \qquad \text{if } i \in A$$

$$c_{i+1} = a_{i+1}(z) \cdot (z - t) + a_{i+1}(z) \cdot \frac{\Omega_{i+1}(t)}{\Omega_i(t)} - a_i(z) \qquad \text{if } i \in U.$$

Consequently, for $j \in \{0, ..., 2p\}$, $t, z \in \mathbb{C}$, there holds

$$\begin{aligned}
a_j(t) \cdot \Omega_j(t) &= 1 + \sum_{i=0}^{j-1} (a_{i+1}(t) \cdot \Omega_{i+1}(t) - a_i(t) \cdot \Omega_i(t)) \\
&= 1 - \sum_{i \in A,\, i<j} c_i \cdot \Omega_{i+1}(t) + \sum_{i \in U,\, i<j} c_{i+1} \cdot \Omega_i(t) \\
&= 1 - \sum_{i \in A,\, i<j} \{a_i(z) \cdot (z-t) \cdot \Omega_{i+1}(t) + a_i(z) \cdot \Omega_i(t) \\
&\quad - a_{i+1}(z) \cdot \Omega_{i+1}(t)\} + \sum_{i \in U,\, i<j} \{a_{i+1}(z) \cdot (z-t) \cdot \Omega_i(t) \\
&\quad + a_{i+1}(z) \cdot \Omega_{i+1}(t) - a_i(z) \cdot \Omega_i(t)\} \\
&= (t-z) \cdot \left\{ \sum_{i \in A,\, i<j} a_i(z) \cdot \Omega_{i+1}(t) - \sum_{i \in U,\, i<j} a_{i+1}(z) \cdot \Omega_i(t) \right\} \\
&\quad + a_j(z) \cdot \Omega_j(t)
\end{aligned}$$

and hence (16). ∎

LEMMA 3. *$(W_j \mid j \in U)$, as given in (11), are linearly independent polynomials.*

Proof. Given $j \in U$, let the linear functional L_j be the divided difference with respect to the knots $(z_{m+n+i} \mid i \in U, i \leqslant j)$. Then, the *Vandermonde matrix*

$$V := (L_j(W_k))_{j \in U}^{k \in U}$$

is lower triangular having the diagonal entries

$$L_j(W_j) = \prod_{i \in A,\, i \geqslant j+2} (z_{m+n+j} - z_{m+n+i}), \qquad j \in U.$$

Since an unattainable point cannot later become an attainable point, we obtain $L_j(W_j) \neq 0$. Therefore, V is regular and $(W_j \mid j \in U)$ are linearly independent. ∎

5. THE SQUARE BLOCK CASE

In this section we consider the square block case which is characterized by $A = \{0, 1, ..., p-1\}$ and $U = \{p, ..., 2p-1\}$. Then, as seen above, the generalized cross-rule becomes Cordellier's identity for the Padé-table. On the other hand it is known that by reordering the knots we are always able to deal with the square block case; see Section 6.

To apply (9), compute for any $t \in \mathbb{C}$

$$w_j(t) = \prod_{i=j}^{p-1} (t - z_{m+n+i}) \qquad \text{if} \quad j \in \{1, ..., p\}$$

$$w_j(t) = \prod_{i=p}^{j-1} (t - z_{m+n+i}) \qquad \text{if} \quad j \in \{p, ..., 2p-1\}$$

and note that $w_1, ..., w_{2p-1}$ are monic polynomials of degree $p-1, ..., 0, ..., p-1$, respectively. Moreover, (9) states that $(a_j(z) \mid j \in A)$ and $(a_j(z) \mid j-1 \in U)$ are just coefficients of (finite) Newton series. Therefore, by evaluation of the divided difference with respect to the knots $z_{m+n+p}, ..., z_{m+n+j-1}$ and to the variable t in (9) we easily obtain the following corollary.

COROLLARY 1. *In the square block case, i.e.,* $A = \{0, 1, ..., p-1\}$ *and* $U = \{p, ..., 2p-1\}$, *we have for any* $j \in \{p+1, ..., 2p\}$

$$a_j(z) = \sum_{i=0}^{2p-j} a_i(z) \cdot [z_{m+n+p}, ..., z_{m+n+j-1}]\, \omega_{m+n+i+1, m+n+p}.$$

As already studied in Example 2 in Section 3, for Padé approximation $(z_\nu = 0)$ Corollary 1 reduces to $a_j(z) = a_{2p-j}(z)$. For arbitrary knots and a singular block with the shape of a square, $a_j(z)$ depends just on $a_0(z), ..., a_{2p-j}(z)$, which is illustrated in Table III. It is the aim of the following sections to investigate this behavior in the general case.

Remark. If we consider the *meromorphic functions* NE_j and SW_j and the *polynomials* a_j instead of their *values* at a fixed complex number z, the relations mentioned in Corollary 1 and Table III can be written more

TABLE III

r	$n-1$	n					
$m-1$		NE_0	NE_1	$\cdots$	NE_{2p-j} ↘		
m	SW_0	C	$\cdots$	C			
	SW_1	$\vdots$	$\cdot$				NE_j
	$\vdots$	C				C	
	SW_{2p-j} ↘				$\cdot$	$\vdots$	
				C	$\cdots$	C	
			SW_j				

compactly. Indeed, using Theorem 2, for $0 \leqslant i < j \leqslant p$, $t, z \in \mathbb{C}$ and the square block case there holds

$$[t, z]\, a_j = \sum_{v=0}^{j-1} a_v(z) \cdot \frac{\Omega_{v+1}(t)}{\Omega_j(t)} = \sum_{v=0}^{j-1} a_v(z) \cdot \omega_{m+n+v+1,\, m+n+j}(t).$$

Evaluating this identity in the variable t for divided difference with respect to the knots $z_{m+n+i}, ..., z_{m+n+j-1}$ gives

$$a_i(z) = [z_{m+n+i}, ..., z_{m+n+j-1}, z]\, a_j, \qquad 0 \leqslant i < j \leqslant p.$$

Consequently, if a_j is known a_i can be computed as a certain divided difference.

Therefore under the assumptions of Corollary 1, $j \in \{p+1, ..., 2p\}$, we have

$$\begin{aligned} a_j(z) = & \sum_{i=0}^{2p-j} [z_{m+n+i}, ..., z_{m+n+2p-j-1}, z]\, a_{2p-j} \\ & \cdot [z_{m+n+p}, ..., z_{m+n+j-1}]\, \omega_{m+n+i+1,\, m+n+p} \\ = & \sum_{i=0}^{2p-j} [z_{m+n+p}, ..., z_{m+n+j-1}, z_{m+n}, ..., z_{m+n+i}]\, \omega_{m+n,\, m+n+p} \\ & \cdot [z_{m+n+i}, ..., z_{m+n+2p-j-1}, z]\, a_{2p-j} \\ = & [z_{m+n+p}, ..., z_{m+n+j-1}, z_{m+n}, ..., z_{m+n+2p-j-1}, z] \\ & \cdot (\omega_{m+n,\, m+n+p} \cdot a_{2p-j}), \end{aligned}$$

the last equality following by Leibniz' rule, where we used $\deg a_{2p-j} = 2p-j$. This yields

$$a_j(z) = [z_{m+n+p}, ..., z_{m+n+j-1}, z](\omega_{m+n+2p-j,\, m+n+p} \cdot a_{2p-j}). \qquad (18)$$

Hence—as for Padé approximants—in the square block case the *polynomial* a_j can be computed if we only know the *polynomial* a_{2p-j} (cf. [9]).

6. Reordering

In this section we derive some identities by reordering to get the square block case of the previous section. The technique of *reordering* is frequently used in the literature (see, e.g., [12]). It is based on the fact that the Newton–Padé approximant $r_{m,n}$ does not depend on the order of the knots $(z_0, ..., z_{m+n})$ (thus they may be permuted) and that $r_{m,n}$ does not depend on $(z_{m+n+1}, z_{m+n+2}, ...)$ (thus they may be changed arbitarily). We stress that, in general, a reordering will change the structure as well as the property of being an attainable or an unattainable point. On the other hand, since an unattainable point cannot become an attainable point later, cf. Remark (ii) in Section 2, a reordering such that only attainable points get a former position does not change attainability.

We return to the definitions of the singular block in the general case of Section 2. In addition, we describe attainable and unattainable points more explicitly by

$$(z_1^A, ..., z_p^A) := (z_{m+n+i} \mid i \in A)$$
$$(z_1^U, ..., z_p^U) := (z_{m+n+i} \mid i \in U),$$

whence

$$z_{m+n+j} = z^A_{l_j+k_j+1} \quad \text{if} \quad j \in A \qquad \text{and} \qquad z_{m+n+j} = z^U_{l_j+1} \quad \text{if} \quad j \in U.$$

Similarly as $\omega_{\mu,\nu}$, define

$$\omega^A_{\mu,\nu}(t) := \prod_{i=\mu}^{\nu-1} (t - z_i^A) \qquad \text{and} \qquad \omega^U_{\mu,\nu}(t) := \prod_{i=\mu}^{\nu-1} (t - z_i^U)$$

for $1 \leqslant \nu \leqslant \mu \leqslant p+1$, $t \in \mathbb{C}$. Note that

$$\Omega_j = \frac{\omega^U_{1,l_j+1}}{\omega^A_{1,k_j+l_j+1}} \qquad \text{and} \qquad w_j = \omega^U_{1,l_j+1} \cdot \omega^A_{k_j+l_j+1,p+1}.$$

Then, we reorder the knots $(z_0, ..., z_{m+n+2p-1})$ into

$$(\bar{z}_0, ..., \bar{z}_{m+n+2p-1}) := (z_0, ..., z_{m+n-1}, z_1^A, ..., z_p^A, z_1^U, ..., z_p^U).$$

Using the data f and $(\bar{z}_0, ..., \bar{z}_{m+n+2p-1})$, we obtain a second Newton–Padé table whose entries are now denoted by $\bar{r}_{\mu,\nu}$. Indeed, this second Newton–Padé table coincides with the original one in each antidiagonal no. $0, ..., m+n-1$, $m+n+2p-1$, $m+n+2p$, ... and the singular block is a square, i.e., $\bar{A} = \{0, ..., p-1\}$ and $\bar{U} = \{p, ..., 2p-1\}$. As in the previous sections we define polynomials $\bar{a}_0, ..., \bar{a}_{2p}$. The following theorem determines the connection between a_j and $\bar{a}_j$.

THEOREM 4. *For any $j \in \{0, ..., 2p\}$ there holds $a_j = \bar{a}_j$ if $l_j = 0$ and otherwise*

$$a_j = \sum_{i=0}^{k_j} \bar{a}_i \cdot [z_1^U, ..., z_{l_j}^U] \, \omega^A_{i+2,l_j+k_j+1}. \tag{19}$$

Proof. a_j depends on f and $(z_0, ..., z_{m+n+j-1})$. We consider a second reordering

$$(\tilde{z}_0, ..., \tilde{z}_{m+n+j-1}, \tilde{z}_{m+n+j}, \tilde{z}_{m+n+j+1}, ...)$$
$$:= (z_0, ..., z_{m+n-1}, z_1^A, ..., z_{k_j+l_j}^A, z_1^U, ..., z_{l_j}^U, z_{l_j}^U, z_{l_j}^U, ...)$$

and, analogously, we write $\tilde{r}_{\mu,\nu}$ and $\tilde{a}_0, ..., \tilde{a}_j$ for the Newton–Padé approximants and the related polynomials. Since Newton–Padé approximants are independent of permutation of knots we achieve

$$\bar{a}_i = \tilde{a}_i \qquad \text{for any} \quad i \in \{0, ..., k_j + l_j\}$$

and

$$\tilde{a}_j = a_j.$$

Since $z_{l_j}^U$ is unattainable and then will never become an attainable point later, the singular block of the considered third non-normal Newton–Padé table is a square. Therefore, we may apply Corollary 1 to this third singular block with $\tilde{p} = k_j + l_j$ and obtain

$$\tilde{a}_j = \sum_{i=0}^{2\tilde{p}-j} \tilde{a}_i \cdot [\tilde{z}_{m+n+\tilde{p}}, ..., \tilde{z}_{m+n+j-1}]\, \tilde{\omega}_{m+n+i+1, m+n+\tilde{p}}.$$

Since $2\tilde{p} - j = k_j$ and $\tilde{\omega}_{m+n+i+1, m+n+\tilde{p}} = \omega_{i+2, l_j+k_j+1}^A$, we conclude (19). ∎

Remark. The imaginary polynomials $\bar{a}_0, ..., \bar{a}_{2p}$ are defined by reordering. Although reordering could be a numerical tool, we stress here a fictitious reordering which will not be carried out explicitly.

Proof of Theorem 2. Given the data described in (12), by applying (19) we are able to compute successively $\bar{a}_j$ for $j = 0, ..., \max\{k_i \mid i = 0, ..., 2p\} - 1$ and therefore the unknown values $a_j(z)$. ∎

7. Global Identities

In this section we introduce some global rules (illustrated in Figs. 2 and 3) which depend strongly on the structure of the singular block. As already discussed for the square block case, we are most interested in relations between the values $a_0(z), ..., a_{2p}(z)$.

Recall from Theorem 2 that we can expect only rules combining $a_j(z)$ with $a_0(z), ..., a_{j-1}(z)$ if

$$k_j \leqslant \max\{k_0, ..., k_{j-1}\},$$

which is obviously equivalent to

$$j-1\in U \qquad \text{or} \qquad \exists i<j \qquad \text{with} \quad k_i=k_j.$$

Therefore, Corollary 2 and 3 discuss the cases $j-1\in U$ and $k_i=k_j$, respectively, which are consequences of Theorem 3. The most interesting special cases are stated explicitly.

COROLLARY 2 *For $j-1\in U$ let $x_1, ..., x_q\in\mathbb{C}$ denote zeros of w_{j-1}, counting multiplicities. Then, for $z\in\mathbb{C}$, there holds*

$$a_j(z)\cdot[x_1, ..., x_q, z^U_{l_j}]\,w_{j-1} = \sum_{i\in A,\, i<j-1} a_i(z)\cdot[x_1, ..., x_q, z^U_{l_j}]\,w_{i+1} - \sum_{i\in U,\, i<j-1} a_{i+1}(z)\cdot[x_1, ..., x_q, z^U_{l_j}]\,w_i. \tag{20}$$

Proof. By Theorem 3, we obtain after multiplication with $\prod_{i\in A}(t-z_{m+n+i})$

$$w_{j-1}(t)\cdot[t, z]\,a_{j-1} = \sum_{i\in A,\, i<j-1} a_i(z)\cdot w_{i+1}(t) - \sum_{i\in U,\, i<j-1} a_{i+1}(z)\cdot w_i(t).$$

This equation is evaluated for $[x_1, ..., x_q, z^U_{l_j}]$ with respect to t. Since $j-1\in U$, Lemma 2 leads to

$$[z, z^U_{l_j}]\,a_{j-1} = a_j(z),$$

and this concludes the proof. ∎

EXAMPLE 4. Taking $q=0$ in Corollary 2 yields a simple rule for $a_j(z)$ provided that $w_{j-1}(z^U_{l_j})\neq 0$. This assumption can be dropped if we chose $(x_1, ..., x_q)=(z^U_{l_j}, ..., z^U_{l_j})$ with q being the multiplicity of the zero $z^U_{l_j}$ of w_{j-1}.

Throughout this section, we define for $i, j\in\{0, ..., 2p\}$

$$d_{i,j} := [z^U_1, ..., z^U_{l_j}, z^A_{k_j+l_j+1}, ..., z^A_p]\,w_i \tag{21}$$

such that $d_{i,j}=1$ if $k_i=k_j+1$ and $d_{i,j}=0$ if $k_i>k_j+1$. Moreover,

$$d_{i,j}=\begin{cases} [z^U_{l_i+1}, ..., z^U_{l_j}]\,\omega^A_{k_i+l_i+1,\, k_j+l_j+1} \\ \qquad \text{if} \quad i<j \quad \text{and} \quad l_i<l_j \\ 0 \quad \text{if} \quad i\leqslant j \quad \text{and} \quad l_i=l_j \\ [z^A_{k_j+l_j+1}, ..., z^A_{k_i+l_i}]\,\omega^U_{l_j+1,\, l_i+1} \\ \qquad \text{if} \quad i>j \quad \text{and} \quad k_i+l_i>k_j+l_j \\ 0 \quad \text{if} \quad i\geqslant j \quad \text{and} \quad k_i+l_i=k_j+l_j \end{cases}$$

EXAMPLE 5. Let us take $(x_1, ..., x_q) = (z_1^U, ..., z_{l_j-1}^U, z_{k_j+l_j+1}^A, ..., z_p^A)$ in Corollary 2, then some summands vanish in (20). We obtain for $j-1 \in U$, $z \in \mathbb{C}$

$$a_j(z) = \sum_{\substack{i \in A,\, i < j-1 \\ k_i \leqslant k_j}} a_i(z) \cdot d_{i+1,j} - \sum_{\substack{i \in U,\, i < j-1 \\ k_{i+1} \leqslant k_j}} a_{i+1}(z) \cdot d_{i,j}. \tag{22}$$

This identity coincides for the square block case with Corollary 1. Using (22) we will show the simpler identity (25) in Example 7 below.

The following rules combine $a_i(z), ..., a_j(z)$ if a_i and a_j have the same degree.

COROLLARY 3. *Let* $i, j \in \{0, ..., 2p\}$, $i < j$ and $k_i = k_j$. *For* z, $x_1, ..., x_{p-k_j} \in \mathbb{C}$, *there holds*

$$a_j(z) - a_i(z) = \sum_{v \in U,\, i \leqslant v < j} a_{v+1}(z) \cdot [x_1, ..., x_{p-k_j}]\, w_v$$
$$- \sum_{v \in A,\, i \leqslant v < j} a_v(z) \cdot [x_1, ..., x_{p-k_j}]\, w_{v+1}. \tag{23}$$

Proof. Setting $h(t) := a_j(t) \cdot w_j(t) - a_i(t) \cdot w_i(t)$, Theorem 3 gives after multiplication with $(t-z)$

$$h(t) = w_j(t) \cdot a_j(z) - w_i(t) \cdot a_i(z)$$
$$+ \sum_{v \in A,\, i \leqslant v < j} a_v(z) \cdot w_{v+1}(t) \cdot (t-z)$$
$$- \sum_{v \in U,\, i \leqslant v < j} a_{v+1}(z) \cdot w_v(t) \cdot (t-z).$$

Since $\deg w_j = \deg w_i = p - k_j$ it suffices to prove $[x_1, ..., x_{p-k_j}, z]\, h = 0$. By definition of $h(t)$, using the methods of the proof of Theorem 3 (cf. (17)), we get

$$h(t) = \sum_{v=i}^{j-1} (w_{v+1}(t) \cdot a_{v+1}(t) - w_v(t) \cdot a_v(t))$$
$$= \sum_{v \in U,\, i \leqslant v < j} c_{v+1} \cdot w_v(t) - \sum_{v \in A,\, i \leqslant v < j} c_v \cdot w_{v+1}(t).$$

Hence by a reordering of the knots $(z_{m+n+i}, ..., z_{m+n+j-1})$ similar to the proof of Theorem 4, we can conclude that $\deg h < p - k_j$. ∎

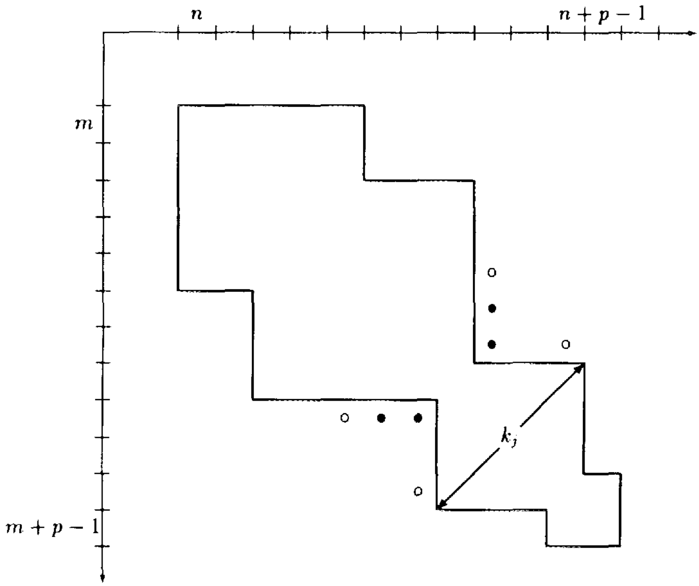

FIGURE 2

EXAMPLE 6. Taking $(x_1, ..., x_{p-k_j}) = (z_1^U, ..., z_{l_j}^U, z_{k_j+l_j+1}^A, ..., z_p^A)$ and using the definition of $d_{i,j}$ in (21), Corollary 3 reads as follows: for $i, j \in \{0, ..., 2p\}$, $i < j$, $k_i = k_j$, $z \in \mathbb{C}$

$$a_j(z) - a_i(z) = \sum_{\substack{v \in U,\, i \leq v < j \\ k_{v+1} \leq k_j}} a_{v+1}(z) \cdot d_{v,j} - \sum_{\substack{v \in A,\, i \leq v < j \\ k_v \leq k_j}} a_v(z) \cdot d_{v+1,j}. \quad (24)$$

If $j-1 \in A$ then $a_j(z)$ does not arise on the right hand side of (24) and hence can be computed using (24). This is illustrated in Fig. 2 where, as above, the singular block is hatched and "○" and "●" mark the position of approximants which arise in (24) on the left and right hand side, respectively. With $i = j-4$ and A, U given by the hatched structure, the relation from Fig. 2 reads

$$a_j(z) - a_{j-4}(z) = a_{j-3}(z) \cdot d_{j-4,j} + a_{j-2}(z) \cdot d_{j-3,j} - a_{j-2}(z) \cdot 0 - a_{j-1}(z) \cdot 0.$$

Note that various i can be chosen. The "most efficient relation" is obtained choosing $i := \max\{v < j \mid k_j = k_v\}$ such that $i \in U$.

EXAMPLE 7. Finally, let us have a closer look at the case $j-1 \in U$. Then, $a_j(z)$ appears on both sides of (24) and hence (24) cannot be used

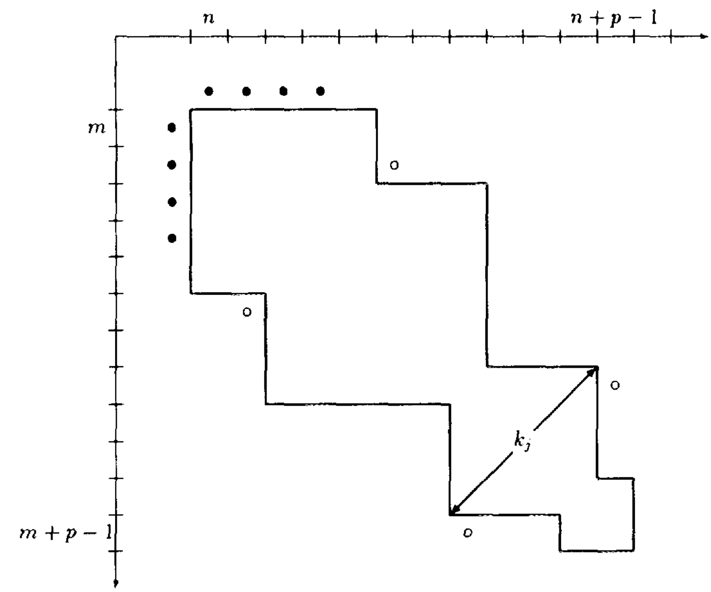

FIGURE 3

to compute $a_j(z)$ from $a_i(z), ..., a_{j-1}(z)$. On the other hand, we can add this rule to (22) and obtain for $j-1 \in U$, $i<j$ and $k_i = k_j$

$$a_j(z) - a_i(z) = \sum_{\substack{\nu \in A,\, \nu < i \\ k_\nu \leq k_i}} a_\nu(z) \cdot d_{\nu+1,j} - \sum_{\substack{\nu \in U,\, \nu < i \\ k_{\nu+1} \leq k_i}} a_{\nu+1}(z) \cdot d_{\nu,j}. \qquad (25)$$

These approximants are illustrated in Fig. 3 where, as above, the singular block is hatched, "○" and "●" mark the position of approximants which arise in (25) on the left and right hand side, respectively. Note that the "most efficient rule" is obtained choosing $i = \min\{\nu \mid k_\nu = k_j\}$.

REFERENCES

1. G. A. BAKER AND P. GRAVES-MORRIS, "Padé Approximants, Part I and II," Addison–Wesely, Reading, MA, 1981.
2. B. BECKERMANN, The structure of the singular solution table of the *M*-Padé approximation problem, *J. Comput. Appl. Math.* **32** (1990), 3–15.
3. B. BECKERMANN AND C. CARSTENSEN, A reliable modification of the cross rule for rational Hermite interpolation, *Numer. Algorithms* **3** (1992), 23–44.
4. G. CLAESSENS, On the Newton–Padé approximation problem, *J. Approx. Theory* **22** (1978), 150–160.

5. G. CLAESSENS, On the structure of the Newton–Padé table, *J. Approx. Theory* **22** (1978), 304–319.
6. G. CLAESSENS, A useful identity for the rational Hermite interpolation table, *Numer. Math.* **29** (1978), 227–231.
7. F. CORDELLIER, Démonstration algébrique de l'extension de l'identité de Wynn aux table Padé non normal, *in* "Padé Approximation and Its Applications" (L. Wuytack, Ed.), Springer-Verlag, Berlin/New York, 1979.
8. M. H. GUTKNECHT, Continued fractions associated with the Newton–Padé table, *Numer. Math.* **56** (1989), 547–589.
9. P. R. GRAVES-MORRIS, Talk at the JCAM conference, Leuven 1984, unpublished.
10. H. E. SALZER, Note on Osculatory Rational Interpolation, *Math. Comput.* **16** (1962), 486–491.
11. D. D. WARNER, "Hermite Interpolation with Rational Functions," Ph.D. thesis, Univ. of California, San Diego, 1974.
12. L. WUYTACK, On the osculatory rational interpolation problem, *Math. Comput.* **29** (1975), 837–843.